Shape from Moments as an Inverse Problem*

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Abstract

We discuss the recovery of a planar polygon from its measured complex moments. Previous work on this problem gave necessary and sufficient conditions for such successful recovery and focused mainly on the case of exact measurements. This paper extends these results by treating the case where a longer than necessary series of noise corrupted moments is given. Similar to methods found in array processing, and system identification we discuss possible estimation procedures. We then present an improvement over these methods based on the direct use of the Maximum-Likelihood estimator. Finally, we show how regularization, and thus Maximum A-posteriori Probability estimator could be applied to reflect prior knowledge about the recovered polygon.

1 Introduction

An intriguing inverse problem proposed in [9] suggests the reconstruction of a planar polygon from a set of its complex moments. Considering an indicator function being 1 in the interior of the polygon and 0 elsewhere, these moments are global functions created by integrating the power function z^k over the plane, and weighted by this indicator function. Given such a finite list of values, the problem posed in [9] is focused on the necessary and sufficient conditions that allow a recovery of the polygon vertices from the given exact moments. In later work, described in [6], the treatment of this reconstruction problem is extended by suggesting better numerical procedures.

When the given moments are contaminated by noise, the recovery problem becomes an estimation one. Previous work on the shape-from-moment problem concentrated on the numerical aspects of the noiseless case. In this work we would like to extend the treatment to a given noisy but longer set of moments. The principal question we are facing is how to robustify the existing procedures to stably recover the polygon vertices from perturbed moment data.

Interestingly, the formulation of the shape-from-moments problem is very similar to other diverse applications such as (i) identifying an auto-regressive system using its output; (ii) decomposing a signal built as a linear mixture of complex exponentials; (iii) estimating the direction of arrival (DOA) in array processing; and more. The literature in these fields offer many algorithms for solving the underlying estimation problem (see [3] for detailed literature survey).

We explore an improvement of the above algorithms by using them to produce an initial solution, and refine it by exploiting the formulation of the problem via Maximum-Likelihood (ML) estimator. Through this change we are also able to incorporate prior knowledge about the desired polygon and use a regularization term. By this we introduce the use of the Maximum A-posteriori Probability (MAP) estimator.

This paper is structured as follows: In the next section we follow [9] formulating the shape-from-moment problem. In Section 3 we describe the Prony's and the Pencil methods. Section 4 presents the improvement over the above algorithms using the ML and the MAP estimation approaches. Simulations and discussion are given in Section 5. Concluding remarks are given in Section 6. We note that a wider and more detailed description of this work can be found in [3].

2 Problem Formulation

An arbitrary closed N-sided planar polygon P is assumed. Its vertices are denoted by $\{z_n\}_{k=1}^{N}$. These values are scalar and complex. Based on Davis's Theorem [1], there exists a set of N coefficients $\{a_n\}_{k=1}^{N}$, depending only on the vertices, such that for any analytic function $f(z)$ in the closure of P, we have

$$\iint_P f''(z)dxdy = \sum_{n=1}^{N} a_n f(z_n). \tag{1}$$

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Davis' Theorem shows that the coefficients $\{a_n\}_{k=1}^N$ are related to the vertices via the equation

$$a_n = \frac{i}{2}\left(\frac{\overline{z}_{n-1} - \overline{z}_n}{z_{n-1} - z_n} - \frac{\overline{z}_n - \overline{z}_{n+1}}{z_n - z_{n+1}}\right). \qquad (2)$$

Since the polygon is closed, we define $\forall k,\ z_{N+k} = z_k$. This formula is exploiting not only the vertices themselves, but also their connection order. For a geometric interpretation of this relationship, see [6, 9].

A special case of interest is obtained for the analytic function $f(z) = z^k$. Using (1) we get

$$k(k-1)\iint_P z^{k-2}dxdy = \sum_{n=1}^{N} a_n f(z_n) = \sum_{n=1}^{N} a_n z_n^k. \quad (3)$$

The expression $\iint_P z^{k-2}dxdy$ stands for the $(k-2)^{th}$ moment computed over the indicator function defined as 1 inside the polygon and zero elsewhere. We denote $k(k-1)\iint_P z^{k-2}dxdy$ as the complex moment τ_k. Clearly, by definition we have that $\tau_0 = \tau_1 = 0$.

Our reconstruction problem is defined as follows: Assume that $M+1$ complex moments, $\{\tau_k\}_{k=0}^M$, are measured and known exactly. How can we recover the polygon vertices using the above relationships? In order to answer this question, we start by forming a set of equations from (3)

$$\begin{bmatrix} \tau_0 \\ \tau_1 \\ \tau_2 \\ \vdots \\ \tau_M \end{bmatrix} = \begin{bmatrix} z_1^0 & z_2^0 & \dots & z_N^0 \\ z_1^1 & z_2^1 & \dots & z_N^1 \\ z_1^2 & z_2^2 & \dots & z_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^M & z_2^M & \dots & z_N^M \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}. \quad (4)$$

Define $\underline{t}_{\{k_1,k_2\}}$ as the column vector of length k_2 containing the complex moments starting with τ_{k_1}. Also, define $\mathbf{V}_{\{k_1,k_2\}}$ as the Vandermonde matrix of size $k_2 \times N$ built from the vertices $\{z_n\}_{n=1}^N$ with powers starting with k_1. Finally, define the vector $\underline{a}$ as a column vector of length N containing the parameters a_n. Then, the above equation can be re-written as

$$\underline{t}_{\{0,M+1\}} = \mathbf{V}_{\{0,M+1\}}\underline{a}. \qquad (5)$$

Both $\mathbf{V}_{\{0,M+1\}}$ and $\underline{a}$ are functions of the vertices. It is interesting to note that in related problems mentioned above, such as AR system identification, decomposition of a mixture of complex exponentials, and the DOA problem, a similar equation is obtained but with coefficients $\{a_n\}_{n=1}^N$ which are independent of the unknown vertices. Nevertheless, the results obtained in this paper will be applicable to these cases as well.

This equation as posed is hard to use for solving for $\{z_n\}_{k=1}^N$ given the complex moments, since it is non-linear as z_n appear both inside the Vandermonde matrix, and are also hidden in the values of a_n. Moreover, solving for $\{z_n\}_{k=1}^N$ using this equation requires not only the vertices but also their order. Alternative relations can be suggested, leading to a practical estimation procedure. These will be discussed in the next section. We further assume that the complex moments are contaminated by additive noise, $\hat{\tau}_k = \tau_k + u_k$, where u_k are assumed to be white i.i.d. zero-mean complex Gaussian noise.

3 Prony and Pencil Methods

One alternative relation to (4) can be suggested, leading to Prony's method [9]. From (4) we see that $\{\tau_k\}_{k=0}^M$ satisfy an N^{th}-order difference equation. It can be shown [9, 6, 3] that the following relationship holds

$$-\begin{bmatrix} \tau_N \\ \tau_{N+1} \\ \tau_{N+2} \\ \vdots \\ \tau_M \end{bmatrix} = \begin{bmatrix} \tau_0 & \tau_1 & \dots & \tau_{N-1} \\ \tau_1 & \tau_2 & \dots & \tau_N \\ \tau_2 & \tau_3 & \dots & \tau_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{M-N} & & \dots & \tau_{M-1} \end{bmatrix} \begin{bmatrix} p_N \\ p_{N-1} \\ \vdots \\ p_1 \end{bmatrix}. \quad (6)$$

The vector $\underline{p}$ denotes a set of coefficients of an N-th order polynomial [9, 6, 3], whose roots are exactly the vertices we seek. Define $\mathbf{T}_{\{k_1,k_2,k_3\}}$ as the Hankel matrix of size $k_2 \times k_3$, built from the complex moments sequence such that the τ_{k_1} is the top left-most entry. Clearly, the matrix uses the moments $\{\tau_k\}_{k=k_1}^{k1+k2+k3-2}$. Using this notation, the above equation is re-written as

$$-\underline{t}_{\{N,M-N+1\}} = \mathbf{T}_{\{0,M-N+1,N\}}\underline{p}. \qquad (7)$$

Note that this equation is true only if the exact moments are used. Using the noisy moments, we should expect to deviate from this relationship. In [9] it is proven that for non-degenerate polygons, the above $(M-N+1)\times N$ matrix $\mathbf{T}_{\{0,M-N+1,N\}}$ is of full rank. Therefore, since we have $M-N+1$ equations and N unknowns, requiring $M \geq 2N-1$ leads to an over-complete and well-posed system of equations.

The polynomial coefficients $\{p_n\}_{n=1}^N$ can now be estimated using (6) in a variety of ways, such as ordinary or total least-squares [9]. Armed with these coefficients, the vertices $\{z_n\}_{n=1}^N$ can be found by computing the roots of the polynomial and this concludes our brief description of the Prony's method.

Going back to (4), we now show another helpful relationship, which leads to the *pencil method* [6]. From the existing $M+1$ equations in this set, take only $M-N+1$, starting with an arbitrary index $0 \leq k \leq N$

and obtain

$$\underline{t}_{\{k,M-N+1\}} = \mathbf{V}_{\{0,M-N+1\}}\text{Diag}\{\underline{a}\}\mathbf{V}^T_{\{k,1\}}. \quad (8)$$

In the above equation we define the operator Diag as the construction of a diagonal matrix from a given vector. By row-concatenation of the N columns corresponding to $0 \leq k \leq N-1$ we get

$$\mathbf{T}_{\{0,M-N+1,N\}} = \mathbf{V}_{\{0,M-N+1\}}\text{Diag}\{\underline{a}\}\mathbf{V}^T_{\{0,N\}}. \quad (9)$$

A similar concatenation could be built with columns corresponding to $1 \leq k \leq N$ resulting in

$$\mathbf{T}_{\{1,M-N+1,N\}} = \quad (10)$$
$$= \mathbf{V}_{\{0,M-N+1\}}\text{Diag}\{\underline{a}\}\text{Diag}\{\underline{z}\}\mathbf{V}^T_{\{0,N\}}.$$

The matrix Diag$\{\underline{z}\}$ is an $N \times N$ diagonal matrix with $\{z_n\}_{n=1}^N$ on its main diagonal. The square $N \times N$ Vandermonde matrix $\mathbf{V}_{\{0,N\}}$ is non-singular since the polygon is assumed to be non-degenerate [6, 9]. Based on (9) and (10) we obtain

$$\mathbf{T}_{\{1,M-N+1,N\}}\mathbf{V}^{-T}_{\{0,N\}} = \quad (11)$$
$$= \mathbf{T}_{\{0,M-N+1,N\}}\mathbf{V}^{-T}_{\{0,N\}}\text{Diag}\{\underline{z}\}.$$

This relationship actually implies that for the pair of matrices $\mathbf{T}_{\{1,M-N+1,N\}}$ and $\mathbf{T}_{\{0,M-N+1,N\}}$, the vertices are their generalized eigenvalues, and the columns of the matrix $\mathbf{V}^{-T}_{\{0,N\}}$ are their generalized eigenvectors. Thus, given the sequence of moments $\{\tau_k\}_{k=0}^M$, we are to form the two $(M-N+1) \times N$ Hankel matrices $\mathbf{T}_{\{1,M-N+1,N\}}$ and $\mathbf{T}_{\{0,M-N+1,N\}}$, and then solve for their generalized eigenvalues using the relation

$$(\mathbf{T}_{\{1,M-N+1,N\}} - \lambda\mathbf{T}_{\{0,M-N+1,N\}})\underline{v} = 0. \quad (12)$$

The eigenvalues are the vertices we desire. A complicating aspect in this relationship is the fact that the obtained pencil is rectangular with more rows than columns. Note that this relationship is true for the noiseless complex moments, and even weak noise added to these moments may lead to no-solution for this system of equations. Solving this pencil problem can be done by squaring the pencil by multiplying both sides from the left by $\mathbf{T}^H_{\{0,M-N+1,N\}}$. It can be shown that this approach is closely related to the Least-Squares Prony method [3]. An alternative, more sophisticated, method is the "Generalized Pencil of Function" (GPOF) method, promoted by Hua and Sarkar [7]. More details about this method and its relation to previously discussed algorithms can be found in [3].

4 Improved Estimation Algorithm

4.1 Exact ML Refinement

Returning to the basic relation in (4), it states $\underline{t}_{\{0,M+1\}} = \mathbf{V}_{\{0,M+1\}}\underline{a}$. In this equation both the matrix $\mathbf{V}_{\{0,M+1\}}$ and vector $\underline{a}$ are functions of the vertices. If the measured moments are contaminated with white Gaussian noise with variance σ_u^2, then using (2), the Maximum Likelihood estimate of the vertices $\{z_n\}_{k=1}^N$ is obtained by

$$\{\hat{z}_n\}_{k=1}^N = \quad (13)$$
$$= \text{ArgMin}_{z_1,\ z_2,\ \ldots\ z_N}\|\hat{\underline{t}}_{\{0,M+1\}} - \mathbf{V}_{\{0,M+1\}}\underline{a}\|^2$$

Using this minimization problem to directly solve for the unknown vertices leads to two difficulties: (i) unless we successfully initialize the optimization procedure, we are bound to fall into a local minimum; and (ii) using this expression calls for the need to solve the problem of ordering the vertices.

As to the first problem, we can assume that one of the above mentioned estimation (either Prony or Pencil based) methods is used and a reasonable estimate of the polygon vertices is indeed given. Thus, using this solution for initial values, we can expect to improve when minimizing, even locally, the above function.

For the problem of ordering, we may consider either solving the ordering directly [2] or disregarding the dependency of the $\underline{a}$ coefficients on the vertices, and replace this vector with the Least-Squares minimizer of this error. Actually, the second approach is suitable for using the proposed refinement idea when dealing with applications such as AR-system identification, where $\underline{a}$ is not a function of the unknowns in any direct way.

Let us define our objective function to be minimized by

$$f(z_1, z_2,\ \ldots\ , z_N) = \quad (14)$$
$$= \sum_{k=0}^{M}\left|\hat{\tau}_k - \sum_{n=1}^{N}\frac{i}{2}\left(\frac{\overline{z}_{n-1} - \overline{z}_n}{z_{n-1} - z_n} - \frac{\overline{z}_n - \overline{z}_{n+1}}{z_n - z_{n+1}}\right)z_n^k\right|^2.$$

Hereafter we assume that, given a proposed solution, we are able to order the vertices properly. Minimizing this function can be done by a line search for each vertex with all the other points fixed. In this way we update the algorithm through a coordinate descent optimization procedure. Alternatively, more sophisticated non-linear least-squares methods could be used.

It is interesting to note that there is a close relationship between the method proposed here and the VarPro method [4] and its variants [8].

4.2 Regularization and MAP Estimator

If we have some prior knowledge about the desired vertices, we can exploit this information and direct the result towards this property by adding a regularization term to (14). As an example, knowing that the angles formed by the vertices are close to 90^o, we may add a term of the form

$$Reg\left(\{z_n\}_{k=1}^N\right) = \qquad (15)$$
$$\sum_{n=1}^N \left(\left| \frac{1}{2} \left(\frac{\overline{z}_{n-1} - \overline{z}_n}{z_{n-1} - z_n} - \frac{\overline{z}_n - \overline{z}_{n+1}}{z_n - z_{n+1}} \right) \right| - 1 \right)^2 .$$

This expression exploits the geometric interpretation of the a_n coefficients having a unit magnitude for vertices forming a 90^o angle. Since the minimization described above is done numerically, any reasonable regularization function can be incorporated and used. When adding this term we should multiply it by some confidence factor λ. Large λ implies that we are confident about this property of the vertices and thus this penalty should play a stronger role. Automatic choice of λ can also be made based on, for instance, the Generalized Cross-Validation (GCV) method [5].

There are many other choices for the regularization function. Just to mention a few, one might be interested in smoothness of the final polygon suggesting

$$Reg\left(\{z_n\}_{k=1}^N\right) = \sum_{n=1}^N |z_n - z_{n-1}|^2$$
or
$$Reg\left(\{z_n\}_{k=1}^N\right) = \sum_{n=1}^N |2z_n - z_{n-1} - z_{n+1}|^2.$$

Alternatively, we might direct the solution to a less "rough" polygon using the fact that the polygon area is given by $\mathcal{A} = 0.5\text{Im}\{\sum_{n=1}^N \overline{z}_n z_{n+1}\}$ and the perimeter is $\Pi = \sum_{n=1}^N |z_n - z_{n-1}|$. Defining $Reg\left(\{z_n\}_{k=1}^N\right) = \Pi^2/\mathcal{A}$ or $Reg\left(\{z_n\}_{k=1}^N\right) = \Pi^2 - 4\pi\mathcal{A}$ we can measure "roughness" and penalize for it.

This regularization idea essentially leads to the Maximum A-posteriori Probability (MAP) estimator. The MAP estimator maximizes the posterior probability of the unknowns given the measurements [3].

5 Results

In this section we present reconstruction results corresponding to the algorithms presented in this paper. We start by creating a polygon and computing its complex moments using Davis's Theorem equations (2) and (4). We then add complex Gaussian white noise to the moments and apply several of the estimation procedures discussed above.

In the first experiment we use a star-shaped polygon with 10 veritces, and apply the GPOF method for obtaining an initialization solution. In this experiment we assumed $\sigma_u = 1e-4$. We then update each vertex only once based on a local coordinate search, i.e., choosing the location corresponding to the nearest local minimum in a search window. Figure 1 shows the behavior of the penalty function per each vertex while fixing all the other vertices. Also overlayed are the location of the true, GPOF estimated (+ symbol), and improved vertices (dot symbol). In this case no regularization was used. The error obtained using the GPOF method is 0.0187, and after the proposed refinement it becomes 0.0074. Table 1 summarizes the results of the average error obtained over 20 runs using the star-shape, applying the GPOF method for initialization, and applying 20 iterations of the coordinate descent algorithm. Each such iteration updates every vertex once, and so we have 200 overall updates. As can be seen, the results are improved dramatically compared to the initialization.

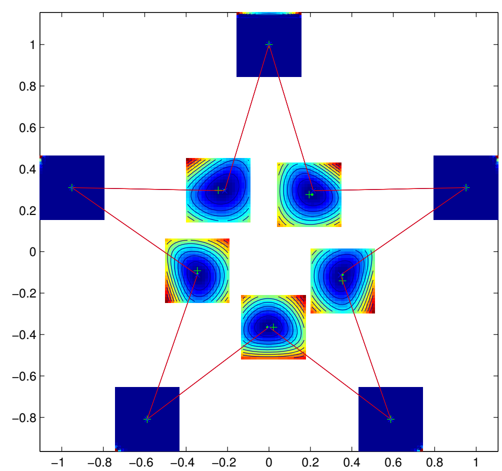

Figure 1: Estimation improvement - the penalty function as a function of each vertex separately.

Noise var.	GPOF	Direct ML
1e-3	3.70e-1	3.19e-1
1e-4	1.49e-2	5.41e-3
1e-5	1.72e-3	2.12e-4

Table 1 - Star-shape - overall RMSE for the GPOF method (used as initialization) and the Direct ML approach.

We next use an 'E'-shaped polygon with 8 vertices, with $\sigma_u = 1e-3$, and we initialize using the LS-Prony algorithm. Similar to the process that created Figure 1, Figure 2 presents the ML function as obtained by perturbing each vertex assuming that all the others are fixed. We chose LS-Prony initialization and high noise variance in order to better see the errors, and the achieved improvement. In this example, the error of the LS-Prony was found to be 0.106. After the improvement stage we obtained an error of 0.062.

Figure 3 shows a similar result when regularization term promoting 90^o angles is added. The regularization coefficient is $\lambda = 1000$. The error is reduced further to 0.041 using a simple and single coordinate descent update. Table 2 summarizes the results obtained for the E-shape using the GPOF method as initialization and 20 iterations of the coordinate descent algorithm, with and without regularization. Again, we averaged 20 experiments in order to see the aggregate effect of the Direct ML algorithm, and the regularization.

Noise var.	GPOF	Direct ML	MAP
1e-3	4.15e-3	2.16e-3	1.64e-3
1e-4	4.04e-4	3.13e-4	2.85e-4
1e-5	4.48e-5	1.23e-5	1.13e-5

Table 2 - E-shape overall RMSE for GPOF as initialization, the Direct ML, and the MAP approaches.

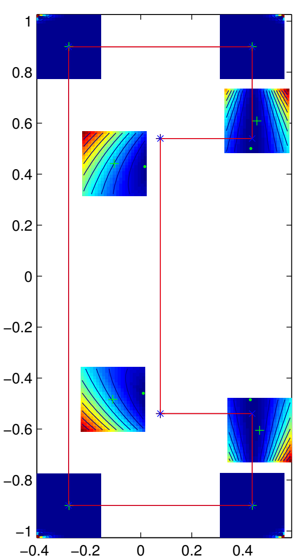

Figure 2: The 'E'-shape estimation improvement - Direct ML approach.

6 Concluding Remarks

This paper discusses the problem of reconstructing a planar polygon from its measured moments. When these moments are contaminated by additive noise, statistical estimation procedures are required. Two families of known estimation algorithms are presented - the Prony and the Pencil–based methods. We use these methods as initialization for direct ML and MAP (via regularization) methods, and show marked improvement over the initial solution.

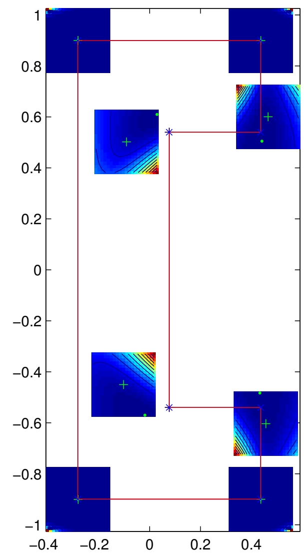

Figure 3: The 'E'-shape estimation improvement - Adding regularization.

References

[1] P. Davis, "Triangle formulas in the complex plane", *Mathematics of Computation*, vol. 18, pp. 569–577, 1964.

[2] S. Durocher, "Graph-theoretic and geometric algorithms associated with moment-based polygon reconstruction", *M.S. Thesis*, Computer-Science Department – The University of British Columbia, August 1999.

[3] M. Elad, P. Milanfar, G. H. Golub, "Shape from Moments - An Estimation Theory Perspective", submitted to the *IEEE Transactions on Signal Processing*, August 2002, Preprint avail. at http://www-sccm.stanford.edu/∼elad.

[4] G. H. Golub, V. Pereyra, "The differentiation of pseudo-inverses and non-linear least squares problems whose variables separate", *SIAM Journal on Numerical Analysis*, vol. 10, no. 2, pp. 413-432, 1973.

[5] G. H. Golub, M. Heath, G. Wahba, "Generalized cross validation as a method for choosing a good ridge parameter", *Technometrics*, vol. 21, pp. 215–224, 1979.

[6] G. H. Golub, P. Milanfar, J. Varah, "A Stable Numerical Method for Inverting Shape from Moments", *SIAM J. on Scientific Comp.*, vol. 21, no. 4, pp. 1222-1243, Dec. 1999.

[7] Y. Hua, T. K. Sarkar, "Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise", *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 38, pp. 814-824, May 1990.

[8] L. Kaufman, V. Pereyra, "A method for separable nonlinear least squares problems with separably nonlinear equality constraints", *SIAM Journal Numer. Anal.* vol. 15, vol. 12–20, 1978.

[9] P. Milanfar, G. Verghese, W. Karl, A. Willsky, "Reconstructing polygons from moments with connections to array processing", *IEEE Transactions on Signal Processing*, vol. 43, no. 2, pp. 432–443, 1995.