Motion From Projection: A Forward Model

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Abstract

One of the most fundamental properties of the Radon (projection) transform is that shifting of the image results in shifted projections. This useful property relates translational *motion in the image to simple displacement in the projections. It is far from clear, however, how more general types of motion in the image domain will be manifested in the projections. In this paper, we will present a model for this phenomenon in the general case; namely, we develop a generalization of the shift property of the Radon transform. We study various properties of the apparent projected motion implied by the model, and study the case of affine motion in particular. We also present illustrative examples, and briefly discuss the* inverse *problem implied by the* forward *model developed herein, along with some possible applications.*

1 Motion in the Projection Domain

The shift property of the Radon transform has found applications in many areas of image processing. For instance, in translational motion estimation from a video sequence [1]. More importantly, projections acquired while the subject undergoes linear motion can be corrected using this property before a reconstruction of the image is attempted.

The shift property of the Radon transform shows that translational motion in the image domain results in translational motion in the projection domain. More specifically, if $g(p,\theta) = \mathcal{R}_\theta[f]$ is the projection of $f(x,y)$ at angle θ defined by

$$g(p,\theta) = \int\int_D f(x,y)\delta(p - x\cos(\theta) - y\sin(\theta))dxdy, \quad (1)$$

we have $\mathcal{R}_\theta[f(x - v_{0x}, y - v_{0y})] = g(p - v_0^T w(\theta), \theta)$, where $v_0 = [v_{0x},\ v_{0y}]^T$ and $w(\theta) = [\cos\theta,\ \sin\theta]^T$ is a unit direction vector.

To the extent that the underlying motion in the image domain can be adequately modeled as translational, this shifting property of the Radon transform is exceedingly useful in application. But more generally, one might naturally wonder what happens in the projection domain if the motion in the image domain is *not* a simple displacement. As an example, respiratory motion during CAT scans can be modeled as a combination of expansion (magnification) and displacement [2]. The shifting property of the Radon transform is no longer adequate in describing the effect of general motion in the image on the projections. Hence a generalization is clearly needed. In this paper, we develop such a generalization and study some of its fundamental implications and properties.

To begin our development of a model for projected motion, we first state two useful differentiation properties of the Radon transform. The first is related to the *Radon transform of derivatives* of a function. Let $L(\partial/\partial x, \partial/\partial y)$ denote a linear differential operator, and write the direction vector $w(\theta) = [w_1,\ w_2]^T$. We have

$$\mathcal{R}_\theta[L\,f] = L(w_1\partial/\partial p,\ w_2\partial/\partial p)\,g(p,w). \quad (2)$$

In particular, if L is a homogeneous polynomial of degree m with constant coefficients, then

$$\mathcal{R}_\theta[L\,f] = L(w)\frac{\partial^m g(p,w)}{\partial p^m}. \quad (3)$$

For instance, a useful corollary is

$$\mathcal{R}_\theta\left[v_0^T\,\nabla f\right] = v_0^T w\,\frac{\partial g(p,w)}{\partial p}. \quad (4)$$

The second property relates to the *derivatives of the Radon transform.* Specifically, for integer k and l,

$$\frac{\partial^{k+l}\,g(p,w)}{\partial w_1^k\ \partial w_2^l} = \left(-\frac{\partial}{\partial p}\right)^{k+l}\mathcal{R}_\theta\left[x^k y^l f(x,y)\right], \quad (5)$$

where it must be kept in mind that when derivatives with respect to components of w are computed, the vector w is initially *not* considered a unit vector. The derivatives may later be evaluated for unit direction vectors.

Now, let us consider an image sequence $f(x, y, t)$, which evolves in time according to the spatially varying motion vector field $v(x, y) = [v_1(x, y),\ v_2(x, y)]^T$. Also, consider its corresponding Radon transform sequence $g(p, \theta, t)$, obtained by computing the Radon transform of f for every fixed t.

For a sufficiently small time increment δt, a first order Taylor series expansion of f is as follows:

$$f(x+v_1\delta t, y+v_2\delta t, t+\delta t) \approx f(x,y,t)+v^T\nabla f\ \delta t+\frac{\partial f}{\partial t}\ \delta t \tag{6}$$

Next, we consider the Radon transform applied to both sides of the above:

$$\begin{aligned} \mathcal{R}_\theta\left[\text{LHS}\right] &\approx \mathcal{R}_\theta\left[f(x,y,t)+v^T\nabla f\ \delta t+\frac{\partial f}{\partial t}\ \delta t\right] \quad (7) \\ &= g(p,\theta,t)+\mathcal{R}_\theta\left[v^T\nabla f\right]\delta t+\frac{\partial g(p,\theta,t)}{\partial t}\delta t \end{aligned}$$

Now *define* the function $u(p, \theta, t)$ (henceforth known as the *projected motion*) by

$$u(p,\theta,t)=\frac{\mathcal{R}_\theta\left[v^T\nabla f(x,y,t)\right]}{\partial g(p,\theta,t)/\partial p}. \tag{8}$$

Clearly, this function is well-defined only when $\partial g(p, \theta, t)/\partial p \neq 0$, and when $f(x, y, t)$ is differentiable. We will discuss these requirements in more depth a bit later. For now, assuming that u is thus well-defined, if we replace its definition into (7), we have

$$\begin{aligned} &\mathcal{R}_\theta\left[f(x+v_1\delta t, y+v_2\delta t, t+\delta t)\right] \approx \\ &g(p,\theta,t)+\ u(p,\theta,t)\frac{\partial g(p,\theta,t)}{\partial p}\delta t+\frac{\partial g(p,\theta,t)}{\partial t}\ \delta t \end{aligned} \tag{9}$$

The right-hand side of (9) now appears quite similar to a Taylor series expansion of $g(p, \theta, t)$. In fact, if $u(p, \theta, t)$ can be replaced by dp/dt, we will have exactly the first-order Taylor series of g on the right-hand side. We can make this substitution only when the differential equation

$$\frac{dp}{dt}=u(p,\theta,t), \tag{10}$$

has a solution, for any fixed θ, over the support of g. The existence and uniqueness theorem for first-order ordinary differential equations [3] states that a unique solution to (10) will exist when $u(p, \theta, t)$ is continuously differentiable (or C^1); that is, $\partial u/\partial p$ must exist and be continuous[1] on a compact subset of the p-axis.

[1]This will imply that u and $\partial u/\partial p$ are also bounded on the same interval.

Referring to the definition of u in (8), we can see that if we require that the vector field v be C^1 and that f be C^2, then $\partial u/\partial p$ exists, it is continuous, and is given by

$$\frac{\partial u}{\partial p}=\frac{\left(\partial\mathcal{R}_\theta[v^T\nabla f]/\partial p\right)\left(\partial g/\partial p\right)-\left(\partial^2 g/\partial p^2\right)\mathcal{R}_\theta[v^T\nabla f]}{\left(\partial g/\partial p\right)^2}. \tag{11}$$

Note that, as before, we have assumed that $\partial g/\partial p \neq 0$. The following proposition, which is the main result of this paper, follows directly from the above definitions and arguments.

Proposition 1 (Projected Motion) *Consider the image sequence $f(x, y, t)$, assumed to be twice continuously differentiable (or C^2), which evolves according to the C^1 vector field $v(x, y)$. Then, for any θ, there exists a C^1 function $u(p, \theta, t)$ such that, to first order,*

$$\mathcal{R}_\theta\left[f(x+v_1\delta t, y+v_2\delta t, t+\delta t)\right] \approx g(p+u\delta t, \theta, t+\delta t), \tag{12}$$

for sufficiently small δt. Furthermore, the function u is given by the identity

$$u(p,\theta,t)\frac{\partial g(p,\theta,t)}{\partial p}=\mathcal{R}_\theta\left[v^T\nabla f(x,y,t)\right] \tag{13}$$

whenever $\partial g/\partial p \neq 0$. We term this relationship the differential **Projected Motion Identity** *(PMI).*

A straightforward corollary of the above result is that under the same assumptions, we have

$$\frac{dg}{dt}=\mathcal{R}_\theta\left[\frac{df}{dt}\right]. \tag{14}$$

That is, locally, the projection of the total derivative of f is the total derivative of the projection of f ($\mathcal{R}_\theta$ and the total derivative operation commute). An immediate consequence is that if the optical flow brightness constraint $df/dt = 0$ is assumed to hold in the image domain, then (14) implies that this constraint also holds in the projection domain: $dg/dt = 0$, with motion in this domain given by (13).

The PMI is a natural generalization of the shift property of the Radon transform and implies the standard shift property if the motion vector is spatially invariant. Furthermore, it is worth noting that as with the classical shift property, the PMI holds in any dimension. That is, if the Radon transform of a scalar function of n real variables is defined as its integrals over hyperplanes of dimension $n - 1$, the arguments presented above would yield the same result except that v would be an n-dimensional vector field.

2 Properties of the Projected Motion

Several interesting properties and implication of the projected motion, and the model in (13) are worth considering. First, we note that u is time-varying even though the vector field v may not be so. This is due to the dependence of u on the gradient of the image, which varies with time. Another observation worth making is that by invoking the directional derivative property (2), we can rewrite $\partial g/\partial p$ in the image domain and express the PMI as follows:

$$u(p,\theta,t)\mathcal{R}_\theta\left[w^T\nabla f\right] = \mathcal{R}_\theta\left[v^T\nabla f\right]. \qquad (15)$$

The insight we gain here is that u is expressible as the *ratio* of two projections; namely, the projection of the directional derivative of the image parallel to v (sometimes called the *advective* derivative of f), and the directional derivative of the image parallel to the unit vector $w(\theta)$, when the latter projection is not zero. Intuitively, at points where $\mathcal{R}_\theta\left[w^T\nabla f\right]$ vanishes, there is no perceived motion in the projection taken at angle θ, and hence, as expected, u is not well defined. It is also interesting to note that in each direction of projection, the correspondence between the vector field v and the function u is not unique. Namely, for a given θ, both v and $v+v_\perp$ yield the same u if $v_\perp$ is such that $\mathcal{R}_\theta[v_\perp^T\nabla f]=0$.

A number of interesting properties of projected motion can be derived directly from the properties of the Radon transform stated earlier and in [4]. For instance, it follows from the linearity of the Radon transform that for a given image f, if u and u' are the projected motions resulting from the vector fields v and v' respectively, then the projected motion field resulting from $av+bv'$ is simply $au+bu'$, where a and b are arbitrary scalars. This, in turn, implies that if a given vector field v is decomposed according to Helmholtz's theorem [5] into its irrotational and solenoidal components as $v=v_I+v_S$, the projected motion field u has a decomposition of the same kind: $u=u_I+u_S$. Other useful properties of u include periodicity: $u(p,\theta+2k\pi,t)=u(p,\theta,t)$, and *anti-symmetry*: $u(p,\theta,t)=-u(-p,\theta+\pi,t)$. Finally, it is well-known ([4, 6]) that the moments of the projections are linearly related to the moments of the image. Of particular interest is the case of zero-th order moments of a function and its Radon transform, which are in fact equal. Applying this result to the PMI, we obtain

$$\iint v^T\nabla f(x,y,t)\,dx\,dy = \int u(p,\theta,t)\,\frac{\partial g}{\partial p}dp, \qquad (16)$$

which states the intuitively pleasing result that projection conserves the average advective derivative of f.

3 Analysis of Affine Motion in the Projection Domain

Any motion field can be locally approximated (to first order) by affine motion. Hence, it is important to consider this class of motions given by

$$v = v_0 + M\begin{bmatrix} x \\ y \end{bmatrix}, \qquad M=\begin{bmatrix} a & b \\ c & d \end{bmatrix}. \qquad (17)$$

where v_0 is a fixed vector denoting translational motion.

To see specifically how affine transformation behaves in the projection domain, we can consider warping an image $f(x,y)$ by such a transformation. Omitting the details of the derivation (See [7] for details), we obtain

$$\left(u - v_0^T w\right)\frac{\partial g}{\partial p} + \mathrm{tr}(M)g + w^T M\begin{bmatrix} \partial g/\partial w_1 \\ \partial g/\partial w_2 \end{bmatrix}\Bigg|_{|w|=1} = 0. \qquad (18)$$

Much can be learned about the general structure of affine motion in the projection domain by considering the representation of images using Hermite polynomials. In particular, consider

$$f(x,y,0)=\sum_{k,l} f_{kl}H_k(x)H_l(y)e^{-x^2-y^2}, \qquad (19)$$

where $\{H_k(x)H_l(y); k,l=0,1,2,\cdots\}$ is the (weighted) orthogonal basis[2] of Hermite polynomials.

It can be shown (see [7] for details) that for this choice of f, and for sufficiently large p,

$$\begin{bmatrix} \partial g/\partial w_1 \\ \partial g/\partial w_2 \end{bmatrix}\Bigg|_{|w|=1} \approx 2p^2 g\, w, \quad \text{and,} \quad \partial g/\partial p \approx -2\,p\,g.$$

Substituting these approximations into (18) and solving for u we obtain the following neat asymptotic expression for u:

$$u \approx v_0^T w + \left(w^T M w\right)p, \qquad (20)$$

which, incidentally, is an affine function of the variable p.

[2] While a shortcoming of this representation is that the basis functions $H_k(x)H_l(y)$ are not compactly supported when real images are, the inclusion of the exponential factor makes this representation somewhat more realistic for image processing.

4 Some Examples of Projected Motion

In this section we present two examples. In the first, analytical expressions for $u(p, \theta, t)$ for a given image sequence and vector field are derived. In the second example, we apply the motion model developed here to a test image sequence and verify that the resulting estimates of the motion in the projection domain are consistent with our proposed model and our intuitive expectations.

4.1 Example 1

Let $f(x, y, t) = \exp\left(-(x - v_1 t)^2 - (y - v_2 t)^2\right)$, and $v(x, y) = [x,\ y]^T$. Skipping the detailed calculations, we obtain

$$u(p, \theta, t) = p + \frac{1}{2p(1-t)^2}, \tag{21}$$

which has (removable) singularities at $p = 0$ and $t = 1$ where $\partial g / \partial p$ vanishes. Intuitively, the singularity at $t = 1$ is a result of the fact that $f(x, y, 1)$ has null gradient. The singularity at $p = 0$ arises because for this value (and for any angle) the images are moving in a perpendicular direction to $w(\theta)$, and hence no motion can be measured in the projections.

4.2 Example 2

In this example, the *diverging trees* image sequence, described in [8], is used to show that the PMI model for motion agrees with actual measurements of motion in the projections. The said image sequence consists of 40 frames, each having 150×150 pixels, obtained as the camera moves along its line of sight toward the scene, resulting in the (known) divergent motion field $v(x, y) \approx 1.1\,[x,\ y]^T$. The 20-th frame, along with some sample motion vectors are shown in Figure 1. Projections of the frames were computed in the row and column directions, and from these, using a Fourier transform-based technique[3] described in [9], the motion in the projections was measured. These estimated values are shown as the solid and dashed curves in Figure 2. The asymptotic model in (20), with $v_0 = [0,\ 0]^T$ and $M = 1.1I$, then implies that the predicted motion in the projections at any angle should be $u(p) \approx 1.1p$. These values are displayed in Figure 2 as circles. It is evident that they generally agree quite well with the directly estimated values while, not surprisingly, the largest errors occur at the center of the plots near the projection of the focus of expansion. Note that the model exhibits a certain degree of robustness to the extent that it is accurate (at least for this simple motion field) even though the images are neither C^2, nor necessarily well-represented by the model (19) in terms of Hermite polynomials.

[3] any other motion estimation algorithm restricted to 1-D can also be used

5 Conclusions and Future Directions

We considered the question of modeling the mapping between motion in an image (or image sequence) and its projections. To this end, we developed a local first order model (the differential Projected Motion Identity) and showed that it produces results that are reasonable and intuitive. We derived some basic properties of projected motion, and studied the effect of affine motion in the projection domain. This analysis revealed that, at least asymptotically, the projected affine motion is itself affine in nature, and that the effect of rotation tends to dissipate as the inverse distance from the vortex in the projection domain, and is hence difficult to measure.

Generally, the PMI can be considered as an indirect measurement equation (or *forward model*) for motion flow in the image domain. This implies an inverse problem. Namely, given measurements of the projections g and their respective motion field u, how do we reconstruct v? Existing reconstruction algorithms [10, 11] can be applied to recover at least the irrotational part of v from u. However, these measurements u contain information about both solenoidal and irrotational components of the motion field v. Questions of existence and uniqueness of solutions, along with numerically well-behaved algorithms for performing the inversion are the subject of current research by the author. This inverse problem has a number of interesting applications. For instance, in motion estimation from video [1], the natural next step would be to ask whether computationally efficient algorithms using projections can be obtained for more general types of motion. As we can see in Section 3 (Equation 20), this appears to be possible in at least the affine case.

A solution to the inverse problem implied by the PMI is useful in any application where it may be difficult or impossible to collect inner-product measurements [10] of a vector field. In these cases, it may be possible instead to measure ordinary line integral projections of the density field, compute motion in these projections using existing motion estimation techniques (applied in one dimension), and attempt to invert for the desired higher-dimensional vector field. This appears to be a promising direction of research that we are currently pursuing. A forthcoming paper will present some of the results of this effort.

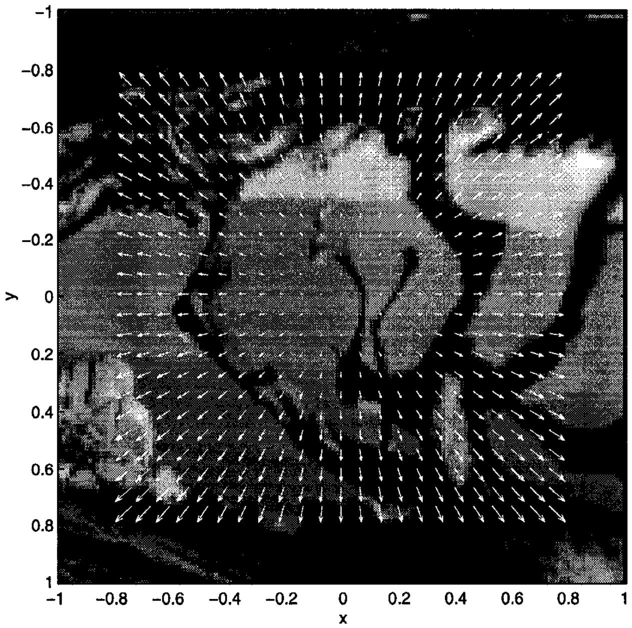

Figure 1: Frame 20 and the optical flow field for Example 2

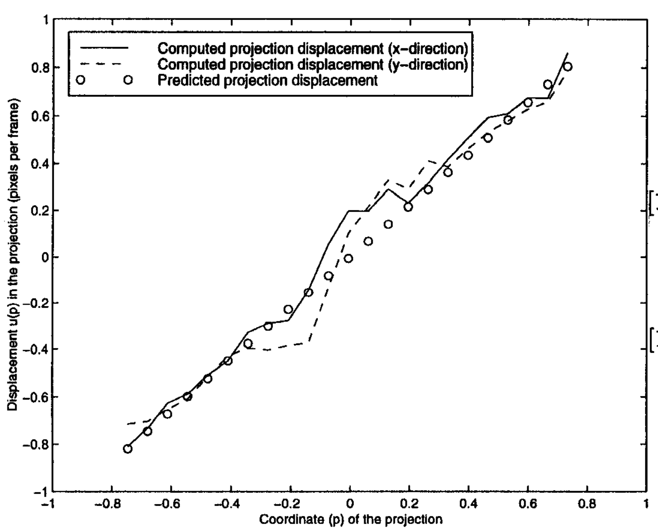

Figure 2: Measured and predicted motion in the projections for Example 2

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