MOMENT-BASED GEOMETRIC IMAGE RECONSTRUCTION

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ABSTRACT

In this paper, we discuss two interesting instantiations of the moment problem in image processing. The first involves the estimation of moments of an image indirectly from projections, and the reconstruction of the image from these moments. The second relates the reconstruction of binary polygons from moments to well-known algorithms in array signal processing. Through these examples, we place the moment problem into a geometric perspective and illustrate how this perspective leads to a number of interesting practical applications in image processing and other fields.

1. INTRODUCTION

The reconstruction of images of objects from indirect measurement has long been of interest in a diverse variety of fields such as medical imaging, machine vision, and oceanographic exploration. The information collected from the particular indirect probe may be in the form of Radon transform projections [2], as is the case in some medical applications, or range (depth) information gathered from a radar return. In many of these scenarios, the gathered data is often used to directly invert the imaging process hence resulting in a rough reconstruction of the underlying object in two or three dimensions. In cases where the available measurements are either sparse or noisy, or often both, the resulting reconstruction is of poor quality due to the *overparameterization* inherent in the direct solution of unconstrained, ill-posed inverse problems such as image reconstruction. To regularize the solution of such ill-posed problems, a-priori information is typically introduced into the reconstruction process in the form of various stochastic priors. These priors, however, seldom contain any information about the *geometry* of the object(s) being sought in the reconstruction. This is in stark contrast to the ultimate goal of most of these imaging tasks; namely, the extraction of geometric information about some object being indirectly probed.

In this paper we describe a set of algorithms that use geometric information in the form of moments, *and* statistical priors simultaneously to solve a class of image reconstruction problems. In particular, we concentrate on two sets of algorithms which have been introduced in [6], [5] and [7], respectively. The first set of algorithms is based on an elementary, but very useful, property of the Radon transform. According to this property, *moments* of an image can be directly estimated from moments of its projections. This result enables us to estimate the moments of an image -which can be thought of as containing geometric information about the image- prior to reconstructing the image and to then use these estimated moments, and their associated error covariances, in conjunction with divergence-based stochastic priors, to regularize the reconstruction process. The second set of algorithms is based on a mathematical result stating that the vertices of any simply-connected planar polygon can be determined from a finite number of its complex moments. In particular, we show that the problem of recovering the vertices of a simply connected polygon from its complex moments can be posed and solved as an *array signal processing* problem.

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2. MOMENT-BASED TOMOGRAPHIC RECONSTRUCTION

The basis of the algorithms discussed here (and introduced in [6, 5, 7]) is the following elementary property of the Radon transform:

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$$\int_{-1}^{1} g(t,\theta)F(t)dt = \int\int_D f(x,y)F(\omega\cdot[x,y]^T)dxdy. \quad (1)$$

where $f(x,y)$ denotes a square-integrable function (image) defined over the unit disk D, $F(t)$ is a square integrable function defined over the interval $[-1,1]$, and $g(t,\theta)$ is the Radon transform [2] of f defined as follows:

$$g(t,\theta) = \int\int_D f(x,y)\delta(t-\omega\cdot[x,y]^T)\,dx\,dy, \quad (2)$$

where $\omega = [\cos(\theta),\ \sin(\theta)]$ and $\delta(\cdot)$ denotes the Dirac delta function.

Although the relationship (1) has been known for quite a long time, little use has been made of it in tomographic reconstruction. In fact, it is well known that by considering $F(t) = e^{-it\omega}$, the celebrated *Projection Slice Theorem* [2] is obtained. What we wish to consider is the case where $F(t)$ is taken to range over a set of orthonormal basis functions over $[-1,1]$. In particular, consider the case when $F(t) = P_k(t)$, where $P_k(t)$ is the k-th order normalized Legendre polynomial over $[-1,1]$. In this basis, Equation (1) relates the moments of the function f linearly to those of its Radon transform. In fact, (1) shows that the k-th order moment of $g(t,\theta)$ is a linear combination of the moments of order k of the function $f(x,y)$. Exploiting this property, we have proved the following result [5, 6]:

Proposition 1 *Given line integral projections of $f(x,y)$ at m different angles θ_j in $[0,\pi)$, one can uniquely determine the first m moment sets (collection of all moments of order m.) This can be done using only the first m orthogonal moments of the projections. Furthermore, moments of $f(x,y)$ of higher order* cannot *be uniquely determined from m projections.*

Given this result, it is straightforward to show that Maximum Likelihood (ML) estimates of the moments of f can be obtained directly from noisy measurements of $g(t,\theta)$. In fact, this turns out to be a *linear* estimation problem. Having obtained these ML moment estimates up thorough order N, along with their associated covariance matrices, we set out to reconstruct the underlying image by defining the I-Divergence Regularization cost functional as

$$J_{IDR}(f,f_0) = \gamma D(f,f_0)+\frac{1}{2}(\mathcal{L}_N(f)-\widehat{\mathcal{L}}_N)^T\Sigma_N(\mathcal{L}_N(f)-\widehat{\mathcal{L}}_N), \quad (3)$$

where $\gamma\in(0,\infty)$ is the regularization parameter, $D(f,f_0)$ denotes the divergence between f and a prior estimate f_0, and $\Sigma_N = Q_N^{-1}$ is the inverse of the error covariance matrix for the estimate of the moments up to order N, $\widehat{\mathcal{L}}_N$. It can be shown [5] that the minimizer of this cost functional, which we are able to efficiently compute, is in fact the statistically optimal Maximum-A-Posteriori (MAP) estimate of f based on the estimated moments and a stochastic prior density

$$P(f) = \frac{1}{c}\exp(-\gamma D(f,f_0)). \quad (4)$$

The minimizer of the above cost functional will turn out to have the following form:

$$\hat{f}_{IDR}(x,y) = f_0(x,y)\exp\left(\Phi_N^T(x,y)C_N\right), \quad (5)$$

where $\Phi_N^T(x,y)$ denotes a vector containing the basis functions (products of Legendre polynomials) used in defining the moments of f. The unknown parameter vector C_N can be obtained efficiently by iteratively solving a set of nonlinear algebraic equations [5].

The solution obtained in (5) can be replaced back into the IDR cost functional (3) as the "new" prior. Repeating this process successively yields an iterative refinement of the IDR algorithm. Formally, beginning with $\widehat{f}_0 = f_0$, we iteratively define

$$\widehat{f}_{k+1} = \arg\min_f J(f,\widehat{f}_k). \quad (6)$$

If (6) is carried to convergence[1], $\widehat{f}_k$ converges to the solution of the following equality constrained problem

$$\min_f D(f,f_0), \quad \text{subject to} \quad \mathcal{L}_N(f) = \widehat{\mathcal{L}}_N^{(c)} \quad (7)$$

where $\widehat{\mathcal{L}}_N^{(c)}$ denotes the projection, defined with respect to the inner product $< l_1, l_2 >_{\Sigma_N} = l_1^T\Sigma_N l_2$, of $\widehat{\mathcal{L}}_N$ onto the range of the operator Ω_N, which denotes the operator mapping a square-integrable function $f\in L^2(D)$, to its Legendre moments up to order N.

The constrained optimization problem (7) is a complex problem to solve directly. In our approach, we have derived an efficient iterative procedure for its solution. In fact, this iterative approach does not require an explicit description of the constraint set Range(Ω_N). This is fortunate since no explicit description of this set is known to exist!

Although similar iterative methods for tomographic reconstruction have been proposed in the past, the distinctive features of our approach are the applications to tomography using the estimated moments and the explicit use of the error covariances for these estimates in forming the penalty function to be minimized. Furthermore, to our knowledge, our specific algebraic solution to the minimization of the IDR functional is also new. In addition, our algorithm provides an explicit mechanism for controlling the degrees of freedom and incorporation of prior geometric information in the reconstructions and hence results in better reconstructions as shown in Figure 1.

[1] local convergence can be guaranteed through appropriate choice of regularization parameters γ_k

3. MOMENT-BASED POLYGON RECONSTRUCTION

Next we present novel algorithms for the reconstruction of binary polygons from their estimated *complex* moments. We show, in fact, that this problem can be formulated as an *array processing* problem. The applications of the algorithms we develop to tomography hence expose a seemingly deep connection between the fields of tomography and array signal processing. This connection implies that a host of numerical algorithms such as MUSIC [11], Min-norm [4], and Prony [8] are now available for application to tomographic reconstruction problems.

Our algorithms are based on the idea that the vertices of a simply-connected polygonal region in the plane are determined by a finite number of its complex moments. This result, in turn is based on a little known theorem called the Motzkin-Schoenberg formula, a generalization of which is due to Davis [1]:

Theorem 1 *[1] Let $z_1, z_2, \cdots, z_n$ designate the vertices of a polygon P. Then we can find constants $a_1, \cdots, a_n$ depending upon $z_1, z_2, \cdots, z_n$, (and the way they are connected) but independent of h, such that for all h analytic in the closure of P,*

$$\int\int_P h''(z)\,dx\,dy = \sum_{j=1}^{n} a_j h(z_j). \qquad (8)$$

If $r \geq n$ and $z_{n+1}, \cdots, z_r$ are additional points distinct from $z_1, \cdots, z_n$, and if there are constants $b_1, \cdots, b_r$ which depend only upon $z_1, \cdots, z_r$ such that

$$\int\int_P h''(z)dxdy = \sum_{j=1}^{r} b_j h(z_j) \qquad (9)$$

for all h analytic in the closure of P, then

$$b_j = a_j, \quad 1 \leq j \leq n, \qquad (10)$$

$$b_j = 0, \quad n+1 \leq j \leq r. \qquad (11)$$

The above result gives a *minimal* representation of the integral of h'' over P in terms of discrete values of h. Furthermore, since we can show that each of the a_j is non-zero for a nondegenerate P, the above representation for arbitrary $h(z)$'s can not be reduced to one involving $h(z)$ at fewer points.

In the above theorem, by letting (I) $h(z) = z^k$ and (II) $f(x,y)$ be the indicator function over a simply-connected polygonal region P of the plane, we get

$$\int\int_P (z^k)''\,dx\,dy = \sum_{j=1}^{n} a_j z_j^k \equiv c_{k-2} \qquad (12)$$

Defining the numbers $\tau_k = k(k-1)c_{k-2}$, which we term *weighted complex moments* (**w-complex moments**), with $\tau_0 = \tau_1 = 0$, we have $\tau_k = \sum_{j=1}^{n} a_j z_j^k$, which is, for every k, a direct relationship between the w-complex moments and the vertices of P. Next we show that by applying Prony's method, the vertices of P are uniquely determined by its w-complex moments τ_k up through order $2n-1$. It is interesting to note that these moments (or in fact *all* w-complex moments of a polygon) are in general not sufficient to uniquely specify the interior of the polygon, even though they *do* uniquely specify the vertices.

The explicit connection between the above and array processing emerges when we consider the general array processing problem of estimating the unknowns c_j and z_j from the measured signals y_k given as follows

$$y_k = \sum_{j=1}^{n} c_j z_j^k + v_k, \quad k = 0, \cdots, N-1 \qquad (13)$$

where, z_j denotes an unknown source, c_j denotes an unknown complex amplitude, and v_k denotes (complex) white noise. Now assume that noisy estimates $\widehat{\tau}_k$ of the w-complex moments of a simply-connected n-gon are given:

$$\widehat{\tau}_k = \sum_{j=1}^{n} a_j z_j^k + w_k. \qquad (14)$$

By comparing this measurement equation to (13), we can see that they have exactly the same form; whereby a vertex of the polygon can be interpreted as a radiating source whose corresponding (complex) amplitude shows how it is connected to the other vertices of the polygon. The general formulation of the array processing problem is therefore nearly the same as the formulation of the reconstruction problem of binary polygonal objects from noisy measurements of their w-complex moments. The main difference is that the coefficients a_j are *not* independent variables but are, in fact, deterministic functions of z_j and the order in which they are connected. Nevertheless, if we treat the a_j as independent unknowns, we *can* directly apply array processing methods and then check to see if the a_j so-determined are in fact consistent with one of the finite number of polygons with vertices given by the extracted values z_j.

A novel application of the concepts and algorithms discussed above can be found in the field of tomographic reconstruction. It is easily shown that the moments τ_k are complex linear combinations of Legendre moments of the underlying image, which as we saw in the previous section, can be estimated directly from the projections. Hence, invoking the "moment-property" of the Radon transform discussed in the previous section, we compute estimates of the w-complex moments of the underlying polygon, and having these, we directly apply array processing algorithms to recover the vertices of the polygon. See Figure 2.

Although the similarity between our formulation of the polygon reconstruction problem and array signal processing is striking, it is equally important to point out that there are distinctive features of the tomography problem that may lead to interesting adaptations and modifications of standard array processing techniques. In particular, there are at least three significant differences between tomography and the array processing problem which we do *not* take advantage of here but which may lead to variations on array processing algorithms with enhanced performance for polygonal reconstruction.

The first we have already mentioned, namely the fact that the coefficients a_j in (14) are deterministic functions of the vertices $z_1, z_2, \cdots, z_n$, and the order in which they are connected. Making optimal use of this information would involve solving a highly nonlinear estimation problem. Secondly, as we have discussed, in the tomographic problem, if we have m projections, we can directly produce estimates of the full set of k^{th} order geometric moments $\mu^{(k)}$ for each $k < m$ and not just the complex moment τ_k, which is a (complex) linear combination of the elements of $\mu^{(k)}$. Thus, in using only the τ_k in our reconstruction, we are not using all of the information extracted from the projections. Finally, as we have noted and as is shown in [5, 6], the error variances in the ML estimates of the moments $\hat{\tau}_k$ are a strong function of k, and in fact increase without bound as a function of the order of these moments [6]. This is in stark contrast to the constant variance assumption typically made for the sensor measurements in standard array processing problems [4, 3, 10, 9]. This, in fact, suggests a line of further investigation in order to adapt standard array processing methods to account for the variation in noise power found in moments estimated from tomographic data.

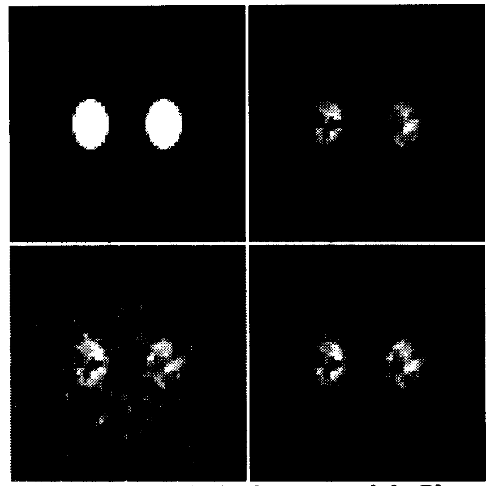

Figure 1: Counter-clockwise from upper left: Phantom, f_0 based on Filtered Back-Projection (% MSE=69.1), It-IDR solution after 3 iter. (% MSE=38.1), It-IDR solution after 10 iter. (% MSE=11.1). Data: 64 proj. w/ 64 samples per proj. SNR = 4.35 dB; moments up to order 8 used.

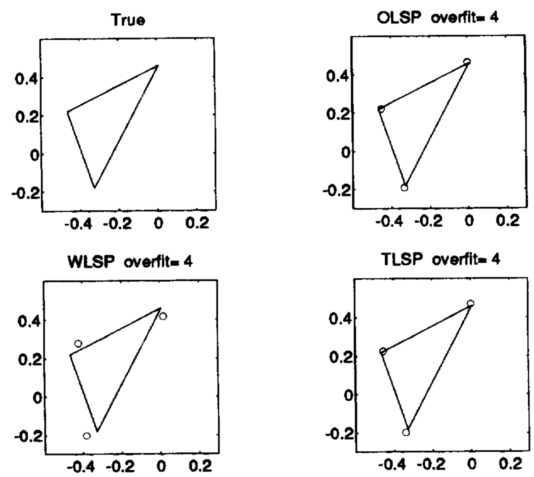

Figure 2: Sample reconstruction at SNR=23.9 dB. Solid: actual, circles: reconstruction. Estimated τ_k used for $0 \leq k \leq 9$

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